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Random walks on combs

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Abstract
We develop techniques to obtain rigorous bounds on the behaviour of random walks on combs. Using these bounds, we calculate exactly the spectral dimension of random combs with infinite teeth at random positions or teeth with random but finite length. We also calculate exactly the spectral dimension of some fixed non-translationally invariant combs. We relate the spectral dimension to the critical exponent of the mass of the two-point function for random walks on random combs, and compute mean displacements as a function of walk duration. We prove that the mean first passage time is generally infinite for combs with anomalous spectral dimension.

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1. Introduction

The fractal structures of random geometrical objects have been under intensive investigation for a number of years, both in connection with quantum gravity [1] and in the study of disordered materials [2–4]. This work is to a large extent aimed at understanding the geometric characteristics of generic objects in the ensembles under study and how these characteristics are reflected in physical phenomena.

An important notion in the study of fractal geometries is the concept of dimension. Definitions of dimension which agree on smooth manifolds do, in general, not do so in the random or fractal case. One important concept is that of *spectral dimension* which is defined to be d_s provided the heat kernel at coinciding points, averaged over the random geometries and viewed as a function of time t, decreases as $t^{-d_s/2}$ as $t \to \infty$. Equivalently, the spectral dimension is a measure of how likely a random walker is to be at the starting point after time t. This notion of dimension is in general different from that of the Hausdorff dimension d_H which

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is defined in terms of the growth of the expectation value of the volume of a geodesic ball of radius r as $r \to \infty$:

$$\langle B(r) \rangle \sim r^{d_H}. \tag{1}$$

We will study in detail many examples of this for the case of random combs in this paper. The discrepancy between these dimensions is also well demonstrated, at least numerically, in quantum gravity.

Early work on the spectral dimension in quantum gravity was done by numerical simulation [5]. This leads to the investigation of random walks on random trees and the spectral dimension of random trees was calculated analytically in [6]; the extension to non-generic trees was given in [7]. Related work on the spectral dimension of trees can be found in [8, 9]. In [10], a scaling relation was derived which relates the spectral dimension to the extrinsic Hausdorff dimension. The paper [11] contains further work on the spectral dimension in two-dimensional gravity. Other applications of random walks to study quantum gravity can be found in [12, 13]. In the condensed matter community the spectral dimension has been investigated for a variety of systems, see, e.g., [2]. While very few exact results have been obtained, random combs in particular have been studied numerically as well as by mean field theory methods. Mean field theory simplifies the problem since it allows one to model the walk on a comb by a random walk on the spine of a comb with a waiting-time distribution which is taken to be the same for all vertices on the spine. The waiting times arise from the excursions that the walk makes into the teeth of the comb. The spectral dimension of random combs with teeth whose lengths obey a power law distribution has been studied in [14], see also [15]. If the exponent of the power law distribution is a then the spectral dimension was found to be given by

$$d_s = \frac{4 - a}{2} \tag{2}$$

for $a \leqslant 2$ but 1 otherwise. In this paper, we prove this result which shows that mean field theory is exact in this case. Mean field theory was shown to be exact in a special case in [16]. We will also show that the spectral dimension of random combs whose teeth may be infinitely long is always $3/2$.

We develop technical tools to prove the above-mentioned results and also apply those techniques to random trees and to some examples of non-random combs. The main new idea is a splitting of random walks as well as random combs into subsets that either yield exponentially suppressed or uniformly controllable contributions to the quantities under consideration (typically d_s). The tools are recursion relations for generating functions, simple monotonicity results and convexity arguments. We believe that the methods can be applied to study random walks on more complicated random graphs.

In the next section, we define the random comb ensembles we wish to study and the most important generating functions and critical exponents. We establish simple monotonicity results and use them to obtain some elementary bounds. In section 3, we study random combs which have an infinitely long tooth at each site on the spine with a non-zero probability. In this case, the spectral dimension is always $3/2$ which is the same as the spectral dimension for a comb with all teeth infinitely long. In section 4, we calculate the spectral dimension of combs with random but finitely long teeth and show that the spectral dimension is determined by the tail of the length distribution. In section 5, we apply our methods to prove upper and lower bounds on the spectral dimension of random trees. In section 6, we calculate the spectral dimension of fixed combs whose tooth length increases along the spine and also combs with infinite teeth whose separation increases along the spine. Section 7 contains results about transport along the backbone of the combs and the full heat kernel on random combs. In the

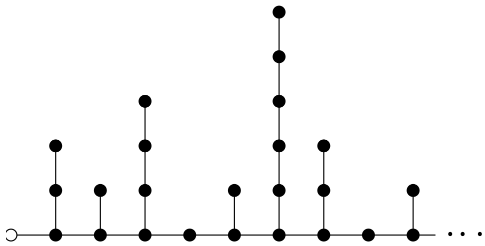

Figure 1. A comb.

final section, we discuss the relevance of our methods and results for random geometry in general and compare our work with relevant results in the mathematics literature. Various technical calculations are relegated to two appendices.

2. Preliminaries

Let N_∞ denote the non-negative integers regarded as a graph so that n has the neighbours $n \pm 1$ except for 0 which only has 1 as a neighbour. Let N_ℓ be the integers $0, 1, \ldots, \ell$ regarded as a graph so that each integer $n \in N_\ell$ has two neighbours $n \pm 1$ except for 0 and ℓ which only have one neighbour, 1 and $\ell - 1$, respectively. A comb C is an infinite rooted tree graph with a special subgraph S called the spine which is isomorphic to N_∞ with the root at 0. At each vertex of S, except the root 0, there may be attached one of the graphs N_ℓ or N_∞. We adopt the convention that these linear graphs which are glued to the spine are attached to their endpoint 0. The linear graphs attached to the spine are called the teeth of the comb, see figure 1. We will find it convenient to say that a vertex on the spine with no tooth has a tooth of length 0. We will denote by T_n the tooth attached to the vertex n on S, and by C_k the comb obtained by removing the links $(0, 1), \ldots, (k - 1, k)$, the teeth $T_1, \ldots, T_k$ and relabelling the remaining vertices on the spine in the obvious way.

2.1. Random walks on combs

We consider simple random walks on the combs. We assume that the walker starts at the root unless we specify otherwise. At each time step the walker steps with equal probabilities to one of the neighbouring vertices. This means that the walker has 1, 2 or at most 3 choices of vertices to step to at any given time and the corresponding probabilities are 1, 1/2 and 1/3, respectively. We are interested in the asymptotic properties of the walk after many time steps. We regard time as integer valued.

Given a comb C, let $p_C(t)$ be the probability that the walker is at the root at time t. Let $p_C^1(t)$ be the probability that the walker is at the root for the first time after $t = 0$ at time t with the convention $p_C^1(0) = 0$. Clearly both $p_C(t)$ and $p_C^1(t)$ vanish unless t is an even number. Given a random walk ω which comes back to the root at time t it is clear that this may be the first return, the second one, etc. We can therefore decompose $p_C(t)$ into a sum over walks that have had a fixed number of intermediate visits to the root before ending there at time t, i.e.,

$$p_C(t) = \delta_{t,0} + \sum_{n=1}^{\infty} \sum_{t_1+t_2+\cdots+t_n=t} \prod_{j=1}^{n} p_C^1(t_j). \qquad (3)$$

We define the generating functions for return to the root and first return to the root by

$$Q_C(z) = \sum_{t=0}^{\infty} z^t p_C(t) \qquad (4)$$

and

$$P_C(z) = \sum_{t=0}^{\infty} z^t p_C^1(t). \tag{5}$$

It follows then from (3) that

$$Q_C(z) = \frac{1}{1 - P_C(z)}. \tag{6}$$

The function $P_C(z)$ is clearly analytic in the unit disc and satisfies $|P_C(z)| < 1$ for $|z| < 1$. Note that functions analogous to P_C and Q_C are defined for the simple random walk on any rooted graph Γ with root of order 1. We will denote these by P_Γ and Q_Γ. If Γ consists of a single vertex, i.e., $\Gamma = N_0$, we adopt the convention that $P_\Gamma(z) = 1$. We will see in a moment that $P_C(1) = 1$ for all combs, i.e., the simple random walk is recurrent on combs.

Consider a fixed comb C. Any walk that contributes to P_C can be decomposed into a first step from 0 to 1; then an arbitrarily large number of round trips into the tooth T_1 intermingled with round trips into the comb C_1 and then a final step from 1 to 0. Each time the walk is located at 1 the probability of stepping into T_1 or C_1 is $1/3$ and likewise the probability of the final step to 0 is $1/3$. It follows that

$$P_C(z) = \frac{z^2}{3 - P_{T_1}(z) - P_{C_1}(z)}. \tag{7}$$

In particular, if C has no tooth at 1, we have

$$P_C(z) = \frac{z^2}{2 - P_{C_1}(z)}. \tag{8}$$

In fact, the recurrence relations (7) and (8) are valid for generalized combs where the teeth are allowed to be arbitrary rooted graphs (with a root of order 1), not necessarily the linear ones that we study here.

Consider now the toothless comb N_∞. The generating function for first return to the root, denoted as P_∞, satisfies

$$P_\infty(z) = \frac{z^2}{2 - P_\infty(z)}. \tag{9}$$

It is convenient to introduce the variable x related to z by

$$1 - x = z^2. \tag{10}$$

The generating functions are even functions of z so they can be regarded as functions of x; we will denote them by the same symbol which should not cause confusion and assume from now on that $0 \leqslant x \leqslant 1$. From (9) we see that

$$P_\infty(x) = 1 - \sqrt{x}. \tag{11}$$

For a finite tooth N_ℓ we denote the generating function for first return of random walks to 0 by P_ℓ. An elementary calculation using the recurrence relation (8) and $P_1(x) = 1 - x$ yields

$$P_\ell(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^\ell - (1-\sqrt{x})^\ell}{(1+\sqrt{x})^\ell + (1-\sqrt{x})^\ell}, \tag{12}$$

see appendix A. We observe that $P_\ell(x)$ is a decreasing function of ℓ for a fixed x and $P_\ell(x) \to P_\infty(x)$ as $\ell \to \infty$.

The comb which has an infinite tooth at each vertex on the spine will be called the *full comb* and denoted by $*$. By (7) the function P_* satisfies

$$P_*(x) = \frac{1 - x}{3 - P_\infty(x) - P_*(x)}. \tag{13}$$

Using (11) and the fact that $P_*(x) \leqslant 1$ we find that

$$P_*(x) = 1 - x^{1/4}\sqrt{1 + \tfrac{5}{4}\sqrt{x} + \tfrac{1}{2}\sqrt{x}}. \tag{14}$$

We define the critical exponent α for a comb C by

$$1 - P_C(x) \sim x^{\alpha}, \qquad \text{as} \quad x \to 0, \tag{15}$$

where $f(x) \sim g(x)$ means that for any $\varepsilon > 0$ there are positive constants c_1 and c_2 such that

$$c_1 x^{\varepsilon} f(x) \leqslant g(x) \leqslant c_2 x^{-\varepsilon} f(x) \tag{16}$$

for $0 < x \leqslant 1$. We see from (11) and (14) that the half line and the full comb have $\alpha = 1/2$ and $\alpha = 1/4$, respectively. It is easy to relate α to the spectral dimension d_s. If

$$p_C(t) \sim t^{-d_s/2} \tag{17}$$

as $t \to \infty$, then

$$Q_C(x) \sim x^{-1+d_s/2} \tag{18}$$

as $x \to 0$ and

$$d_s = 2 - 2\alpha, \tag{19}$$

so the half line and the full comb have spectral dimensions 1 and 3/2, respectively. The value $d_s = 3/2$ for the full comb was first obtained in [17]. In fact, the spectral dimension (if it exists) of any comb lies in the closed interval $[1, 3/2]$. This is a consequence of

$$P_*(x) \leqslant P_C(x) \leqslant P_\infty(x) \tag{20}$$

valid for any comb C. Inequalities (20) follow from the monotonicity lemma below. Furthermore, the lower bound in (20) and (14) imply that random walks on combs are recurrent as claimed above.

We note from (7) that, for fixed x, $P_C(x)$ is a monotonic increasing function of $P_{T_1}(x)$ and $P_{C_1}(x)$. By applying (7) in turn to $P_{C_1}(x), P_{C_2}(x), \ldots, P_{C_{k-1}}(x)$ we find by induction the following result:

Lemma A. *The function $P_C(x)$ is a monotonic increasing function of $P_{T_1}(x), \ldots, P_{T_k}(x)$, $P_{C_k}(x)$ for any $k \geqslant 1$.*

Monotonicity lemma. *$P_C(x)$ is a decreasing function of the length, ℓ_k, of the tooth T_k for any $k \geqslant 1$.*

Proof. By lemma A $P_C(x)$ is a monotonic increasing function of $P_{T_k}(x)$ which is a decreasing function of ℓ_k according to (12). □

Rearrangement lemma. *Let C' be the comb obtained from C by swapping the teeth T_n and T_{n+1}. Then $P_C(x) > P_{C'}(x)$ if and only if $P_{T_n}(x) > P_{T_{n+1}}(x)$.*

Proof. By lemma A, and noting that $T_1, \ldots, T_{n-1}$ are the same for C and C', it suffices to prove that $P_{C_{n-1}} > P_{C'_{n-1}}$ if and only if $P_{T_n} > P_{T_{n+1}}$. By (7) the former holds if and only if $P_{T_n} + P_{C_n} > P_{T_{n+1}} + P_{C'_n}$. It is therefore enough to compute

$$\begin{aligned}\Delta &= P_{T_n} + P_{C_n} - P_{T_{n+1}} - P_{C'_n} \\ &= (P_{T_n} - P_{T_{n+1}})\left(1 - \frac{1-x}{(3 - P_{T_{n+1}} - P_{C_{n+1}})(3 - P_{T_n} - P_{C_{n+1}})}\right),\end{aligned} \tag{21}$$

where we have used (7) and the fact that $C_{n+1} = C'_{n+1}$. We see that $\Delta > 0$ if and only if $P_{T_n} > P_{T_{n+1}}$ which completes the proof. □

2.2. The two-point function

Let C be a comb and let $p^1_C(t;n)$ denote the probability that a random walk that starts at the root 0 at time 0 is at the vertex n on the spine at time t and has not visited the root in the time interval from 0 to t. We will refer to the generating function for these probabilities as the two-point function and denote it by $G_C(x;n)$. Note that $G_C(x;0) = P_C(x)$ and

$$G_C(x;n) = \sum_{t=1}^{\infty}(1-x)^{t/2}p^1_C(t;n). \tag{22}$$

The two-point function can also be expressed as the sum over all random walks from 0 to n which avoid 0 and end at n:

$$G_C(x;n) = \sum_{\omega:0\to n, \omega_t\neq 0 \text{ if } t\neq 0}\ \prod_{t=0}^{|\omega|-1}(\sigma(\omega_t)^{-1}\sqrt{1-x}), \tag{23}$$

where $|\omega|$ is the number of steps in the walk ω and $\sigma(\omega_t)$ is the order of the vertex ω_t where ω is located at time t.

The representation (23) of the two-point function is quite useful and allows us to relate it to the first return generating functions as we now show. Let C be a comb and let C_k be defined as before. If we consider a random walk ω on C which contributes to the two-point function $G_C(x;n)$ we can decompose it into a sequence of n random walks $\omega^1,\ldots,\omega^n$, where ω^1 is a walk from 0 to 1 which is identical to ω until ω leaves the vertex 1 for the last time before going to n, ω^2 is a walk from 1 to 2 which is identical to ω after it left 1 for the last time until it leaves 2 for the last time, etc. The last walk ω^n is the part of ω after it left $n-1$ for the last time and until it ends at n. If for each $k = 1, 2, \ldots, n-1$ we add a last step to ω^k back to the vertex $k-1$ and call the resulting walk $\tilde{\omega}^k$ we see that $\tilde{\omega}^k$ is a walk from $k-1$ to $k-1$ which contributes to $P_{C_{k-1}}$ and any walk contributing to $P_{C_{k-1}}$ can arise in this way. It follows that for $n > 0$

$$G_C(x;n) = \sigma(n)(1-x)^{-n/2}\prod_{k=0}^{n-1}P_{C_k}(x), \tag{24}$$

where $\sigma(n)$ is the degree of the vertex n.

We see from (11) and (24) that the two-point function for the half line, $G_\infty(x;n)$, is given by

$$G_\infty(x;n) = 2\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{n/2} \tag{25}$$

for $n > 0$. We define the mass, $m(x)$, of the two-point function G_C by its rate of exponential decay, i.e.,

$$m(x) = -\lim_{n\to\infty}\frac{\log G_C(x;n)}{n}. \tag{26}$$

For an arbitrary comb, there is no reason to expect the limit (26) to exist but the mass associated with the two-point function for the half line is clearly

$$m_\infty(x) = \frac{1}{2}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}. \tag{27}$$

We can similarly use (14) to compute the two-point function, $G_*(x;n)$ and the mass, $m_*(x)$, for the full comb. It furthermore follows from (20) and (24) that

$$\left(\frac{\sigma(n)}{3}\right)G_*(x;n) \leqslant G_C(x;n) \leqslant \left(\frac{\sigma(n)}{2}\right)G_\infty(x;n) \tag{28}$$

for any comb C. If the mass $m(x)$ exists we define its critical exponent ν by

$$m(x) \sim x^{\nu} \tag{29}$$

as $x \to 0$. It is easy to see that $\nu = 1/2$ for m_∞ and $\nu = 1/4$ for m_*. From (28) we conclude that the critical exponent of the mass for any comb lies in the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$.

The above considerations show that the exponents α and ν coincide for the half line and the full comb. Indeed, we will prove in section 6 that the scaling relation

$$\alpha = \nu \tag{30}$$

holds quite generally.

2.3. Random combs

Let $\mathcal{C}$ denote the collection of all combs. Let $\mathbb{Z}_0^+$ denote the non-negative integers. If we are given a probability measure μ on $\mathbb{Z}_0^+ \cup \{\infty\}$ we can define a probability measure π on $\mathcal{C}$ by letting the length of the teeth be identically and independently distributed by μ. This means that the measure of the set of combs Ω with teeth at $n_1, n_2, \ldots, n_k$ having lengths $\ell_1, \ell_2, \ldots, \ell_k$ is

$$\pi(\Omega) = \prod_{j=1}^{k} \mu(\ell_j). \tag{31}$$

We will refer to the set $\mathcal{C}$ equipped with the probability measure π as a random comb. Measurable subsets $\mathcal{A}$ of $\mathcal{C}$ are called *events* and $\pi(\mathcal{A})$ is the probability of the event $\mathcal{A}$. We define the first return generating functions for random walks on random combs as

$$\bar{P}(x) = \langle P_C(x)\rangle, \tag{32}$$

where $\langle\cdot\rangle$ denotes expectation with respect to the measure π, i.e.,

$$\langle F(C)\rangle = \int F(C)\,\mathrm{d}\pi \tag{33}$$

for any π-integrable function F defined on $\mathcal{C}$. Similarly,

$$\bar{Q}(x) = \langle Q_C(x)\rangle. \tag{34}$$

If $\bar{Q}(x) \sim x^{-1+d_s/2}$ as $x \to 0$ we say that the spectral dimension of the random comb is d_s. Similarly we define the exponent α for the random comb by $1 - \bar{P}(x) \sim x^{\alpha}$. We will see for the examples of random combs studied in this paper that relation (19) holds.

The two-point function of the random comb is defined as

$$\bar{G}(x; n) = \langle G_C(x; n)\rangle. \tag{35}$$

We show below that the mass exists for any random comb. It follows from (28) that this mass $\bar{m}(x)$ satisfies the inequalities

$$m_\infty(x) \leqslant \bar{m}(x) \leqslant m_*(x) \tag{36}$$

and the critical exponent ν of $\bar{m}$ lies in the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$.

2.4. The mass for random combs

In this subsection, we introduce some auxiliary generating functions and prove the existence of the mass for random combs. We assume that we are given a random comb where the lengths of the teeth are identically and independently distributed. For a fixed comb C, define a modified two-point function $G^0_C(x;n)$ by restricting the sum in (23) to walks that stop the first time they hit the vertex n. Then we have the factorization

$$G_C(x;n) = G^0_C(x;n) Q_C(x;n), \tag{37}$$

where $Q_C(x;n)$ is the sum over all walks which begin and end at n and avoid the root 0. Equation (37) can be obtained by considering any walk contributing to the two-point function $G_C(x;n)$ and cutting it at n the first time it hits n. The first part then contributes to $G^0_C(x;n)$ while the second part contributes to $Q_C(x;n)$. Let T_n be the tooth of C at n (which may be empty). Let $P^{(-)}_C(x;n)$ be the generating function for first return of random walks that begin at n, have a first step to $n-1$ and avoid the root. Similarly, let $P^{(+)}_C(x;n)$ be the generating function for first return of random walks that begin at n and have a first step to $n+1$. Then we have

$$Q_C(x;n) = \frac{\sigma(n)}{3 - P^{(-)}_C(x;n) - P^{(+)}_C(x;n) - P_{T_n}(x)}. \tag{38}$$

Using $P^{(+)}_C(x;n) \leqslant 1-\sqrt{x}$, $P^{(-)}_C(x;n) \leqslant 1$ and $P_{T_n}(x) \leqslant 1$ we obtain

$$Q_C(x;n) \leqslant 3x^{-\frac{1}{2}} \tag{39}$$

and hence,

$$G^0_C(x;n) \leqslant G_C(x;n) \leqslant 3x^{-\frac{1}{2}} G^0_C(x;n). \tag{40}$$

Let us define $G^0_C(x;n,n')$ as the sum, analogous to (23), over all walks from n to n' which avoid both n and n' at all intermediate times. Then $G^0_C(x;0,n) = G^0_C(x;n)$. Consider now a walk ω contributing to the two-point function $G^0_C(x;n_1+n_2)$. Cut ω the first time it hits n_1. Cut it again the last time it leaves n_1. Then we obtain three walks, the first of which contributes to $G^0_C(x;n_1)$, the second one starts and ends at n_1, avoiding both 0 and n_1+n_2 and the last one contributes to $G^0_C(x;n_1,n_1+n_2)$. We therefore obtain a factorization:

$$G^0_C(x;n_1+n_2) = G^0_C(x;n_1) R_C(x;n_1) G^0_C(x;n_1,n_1+n_2), \tag{41}$$

where $R_C(x;n) \leqslant Q_C(x;n)$. Hence, by (39),

$$G^0_C(x;n_1+n_2) \leqslant 3x^{-\frac{1}{2}} G^0_C(x;n_1) G^0_C(x;n_1,n_1+n_2). \tag{42}$$

Since the teeth are independently distributed we see that the functions $G^0_C(x;n_1)$ and $G^0_C(x;n_1,n_1+n_2)$ are also independently distributed. We denote the averaged modified two-point functions by $\bar{G}^0(x;n,n')$. Then

$$\bar{G}^0(x;n_1,n_1+n_2) = \bar{G}^0(x;n_2), \tag{43}$$

so

$$\bar{G}^0(x;n_1+n_2) \leqslant 3x^{-\frac{1}{2}} \bar{G}^0(x;n_1) \bar{G}^0(x;n_2), \tag{44}$$

and the function $\log\left(3x^{-\frac{1}{2}}\bar{G}^0(x;n)\right)$ is subadditive in n. It follows by standard arguments that

$$-\lim_{n\to\infty} \frac{\log \bar{G}^0(x;n)}{n} = -\inf_n \frac{\log \bar{G}^0(x;n)}{n}. \tag{45}$$

In view of (40), we conclude that the mass associated with the averaged two-point function $\bar{G}(x;n)$ exists and is given by (45).

2.5. Some bounds

In this subsection, we establish bounds which, for the purpose of calculating α and d_s, allow us to ignore walks that wander too far along the spine. We denote by $E(x)$ a generic non-negative function of $x > 0$ with the property that there are positive constants c_1, c_2 and ε such that

$$E(x) \leqslant c_1 \,\mathrm{e}^{-c_2 x^{-\varepsilon}}. \tag{46}$$

In the following we will let c, c', c_1, c_2, etc denote positive constants whose value may change from line to line.

Let C be a comb and define $P_C^{(n)}(x)$ as the contribution to $P_C(x)$ coming from walks whose maximal distance from the root along the spine is n. Then

$$P_C(x) = \sum_{n=1}^{\infty} P_C^{(n)}(x) \tag{47}$$

and we claim that

$$\sum_{n=N(x)}^{\infty} P_C^{(n)}(x) = E(x) \tag{48}$$

if

$$G_C(x; N(x)) = E(x). \tag{49}$$

This follows from

$$\begin{aligned}\sum_{n=N(x)}^{\infty} P_C^{(n)}(x) &= \sigma(N(x))^{-1} G_C^0(x; N(x))^2 Q_C(x; N(x)) \\ &= \sigma(N(x))^{-1} G_C(x; N(x))^2 Q_C(x; N(x))^{-1},\end{aligned} \tag{50}$$

cf (37), (39) and (40). We conclude, in particular, that if $N(x) \geqslant x^{-\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$, then (48) holds for any C.

Finally, if we have a comb C such that (49) holds for C and that C' is another comb which is identical to C up to the vertex $N(x)$ on the spine, then, by (37) and (40),

$$\begin{aligned}\sum_{n=N(x)}^{\infty} P_{C'}^{(n)}(x) &\leqslant \left(G_{C'}^0(x; N(x))\right)^2 Q_{C'}(x; N(x)) \\ &\leqslant c x^{-\frac{1}{2}} \left(G_C^0(x; N(x))\right)^2 \\ &= E(x).\end{aligned} \tag{51}$$

The estimates (48) and (51) will be used repeatedly in this paper.

3. Combs with infinite teeth at random location

In this section, we consider a random comb for which there is probability $p \in (0, 1)$ that there is an infinite tooth at a vertex on the spine and probability $q = 1 - p$ that there is no tooth, i.e., $\mu(0) = 1 - p$ and $\mu(\infty) = p$ in the notation of subsection 2.3. We will show that in this case the spectral dimension is $3/2$, i.e., the same as for the full comb. It follows immediately that any random comb with a non-zero probability of an infinite tooth at any given vertex has spectral dimension $3/2$.

The strategy of the proof is to use (48) which shows that for a given value of x it suffices to consider walks that do not move beyond a location $N(x)$ on the spine. Then we use the rearrangement lemma to dilute the teeth on the interval from 0 to $N(x) = \left[x^{-\frac{1}{2}-\varepsilon}\right]$ so that they are regularly spaced and we can obtain an upper bound on $\bar{Q}$ which turns out to be of the same form as the trivial lower bound on $\bar{Q}$ coming from comparison with the full comb.

Let L_0 denote the distance from the root to the first (non-trivial) tooth and let L_i, $i \geqslant 1$, denote the distance from the ith tooth to the $(i+1)$th tooth. Since the L_is are independently distributed random variables we see that

$$\pi(\{L_i \leqslant L : i = 0, 1, \ldots, k-1\}) = (1-q^L)^k. \tag{52}$$

If r is a real number we denote its integer part by $[r]$. Fix $\varepsilon \in (0, 1/8)$, choose $k = \left[x^{-\frac{1}{2}-\varepsilon}\right]$ and $L = [x^{-\varepsilon}]$. Let $\mathcal{A}_\varepsilon$ be the event that $L_i > L$ for some $i \in \{0, 1, \ldots, k\}$. Then, by (52), $\pi(\mathcal{A}_\varepsilon) = E(x)$.

Now consider a comb $C \notin \mathcal{A}_\varepsilon$. The spacings between the first k teeth of C are all smaller than or equal to L. By removing all teeth in C except the first k ones and shifting these suitably away from the root we obtain a comb C' whose teeth have constant spacing $L_i = L, i = 0, \ldots, k-1$. Hence, by the monotonicity and rearrangement lemmas we have

$$P_C(x) \leqslant P_{C'}(x) = P_{C'}^{(1)}(x) + P_{C'}^{(2)}(x), \tag{53}$$

where $P_{C'}^{(1)}(x)$ is the contribution to $P_{C'}(x)$ coming from paths which do not pass through the point $\left[x^{-\frac{1}{2}-\varepsilon}\right]$ on the spine and $P_{C'}^{(2)}(x)$ is the remainder. By (48) we have $P_{C'}^{(2)}(x) = E(x)$ uniformly for $C \in \mathcal{C}\backslash\mathcal{A}_\varepsilon$. Moreover, we have

$$P_{C'}^{(1)}(x) = P_{*L}^{(1)}(x) \leqslant P_{*L}(x), \tag{54}$$

where $*L$ is the comb with infinite teeth of spacing L. We conclude from (53) and (54) that

$$P_C(x) \leqslant P_{*L}(x) + E(x). \tag{55}$$

Using the result

$$P_{*L}(x) \leqslant 1 - cx^{\frac{1}{4}+\frac{\varepsilon}{2}} \tag{56}$$

derived in appendix B, we obtain

$$P_C(x) \leqslant 1 - cx^{\frac{1}{4}+\frac{\varepsilon}{2}} \tag{57}$$

and it follows that

$$Q_C(x) \leqslant cx^{-\frac{1}{4}-\frac{\varepsilon}{2}} \tag{58}$$

uniformly for $C \notin \mathcal{A}_\varepsilon$. Hence,

$$\begin{aligned}\bar{Q}(x) &= \int_{\mathcal{A}_\varepsilon} Q_C(x)\,\mathrm{d}\pi(C) + \int_{\mathcal{C}\backslash\mathcal{A}_\varepsilon} Q_C(x)\,\mathrm{d}\pi(C) \\ &\leqslant x^{-\frac{1}{2}}\pi(\mathcal{A}_\varepsilon) + cx^{-\frac{1}{4}-\frac{\varepsilon}{2}}\pi(\mathcal{C}\backslash\mathcal{A}_\varepsilon) \\ &\leqslant cx^{-\frac{1}{4}-\frac{\varepsilon}{2}}.\end{aligned} \tag{59}$$

It follows that $\alpha \leqslant \frac{1}{4} + \frac{\varepsilon}{2}$ and $d_s \geqslant \frac{3}{2} - \varepsilon$ for any $\varepsilon > 0$. In view of the lower bound (20), we obtain

$$\alpha = \tfrac{1}{4}, \qquad d_s = \tfrac{3}{2}. \tag{60}$$

4. Combs with finite random teeth

In this section, we will calculate the spectral dimension of random combs with finite but arbitrarily long teeth. An upper bound on $\bar{P}(x)$ will be obtained by mimicking the argument for the upper bound obtained in the previous section using the fact that if the teeth are sufficiently long they can be replaced by infinitely long teeth up to discrepancies of size $E(x)$. The lower bound will be obtained by a convexity argument.

Let μ be a probability measure on the non-negative integers and set $\mu(\ell) = \mu_\ell$. For simplicity of presentation we assume to begin with that we have a power law distribution $\mu_\ell = c_a \ell^{-a}$, where $a > 1$ since the μ_ℓ sum to 1. Choose $\varepsilon > 0$ and define

$$p(x) = \sum_{\ell=[x^{-\frac{1}{2}-\varepsilon}]}^{\infty} \mu_\ell, \tag{61}$$

for $x > 0$. We shall refer to $p(x)$ as the probability that a tooth is long. Clearly $p(x) \to 0$ as $x \to 0$. More precisely,

$$p(x) \sim x^{g(\varepsilon)}, \tag{62}$$

where

$$g(\varepsilon) = \frac{a-1}{2} + \varepsilon(a-1). \tag{63}$$

Consider now the random comb defined by μ and let $M = [x^{-g(\varepsilon)-\varepsilon}]$ and $K = \left[x^{-\frac{1}{2}-\varepsilon}\right]$. Denote by L_0 the distance from the root to the first long tooth and by L_i, $i \geqslant 1$, the distance from the ith long tooth to the $(i+1)$th long tooth. Let $\mathcal{B}_\varepsilon$ be the event that at least one of the distances $L_0, \ldots, L_{K-1}$ is greater than M. Since the probability that $L_i > M$ is given by

$$(1 - p(x))^M \leqslant \mathrm{e}^{-Mp(x)}, \tag{64}$$

we have

$$\pi(\mathcal{B}_\varepsilon) \leqslant K\mathrm{e}^{-Mp(x)} = E(x). \tag{65}$$

Consider now a comb $C \notin \mathcal{B}_\varepsilon$. By deleting all teeth from C except the first K long teeth and shifting these suitably away from the root we obtain a comb C' whose teeth have a constant spacing $L_i = M$, $i = 0, \ldots, K-1$. By the monotonicity and rearrangement lemmas we have

$$P_C(x) \leqslant P_{C'}(x). \tag{66}$$

Replacing the teeth in C' by infinitely long ones, only changes $P_{C'}(x)$ by $E(x)$. This follows from the fact that walks that go more than a distance $\left[x^{-\frac{1}{2}-\varepsilon}\right]$ into the teeth only contribute $E(x)$ to $P_C(x)$ uniformly in C. By the same argument as the one leading to (55) we obtain

$$P_C(x) \leqslant P_{*M} + E(x), \tag{67}$$

uniformly for $C \notin \mathcal{B}_\varepsilon$.

Let us first consider the case $a < 2$. Then we can choose ε so small that $g(\varepsilon) + \varepsilon < 1/2$. It follows that $M \sim [x^{-\beta}]$ with $0 < \beta < 1/2$ so by (B.2) in appendix B,

$$P_C(x) \leqslant 1 - cx^{\frac{1}{4}+\frac{1}{2}(g(\varepsilon)+\varepsilon)}, \tag{68}$$

for $C \notin \mathcal{B}_\varepsilon$. We conclude from the above and (65) that

$$\langle \bar{P}(x) \rangle \leqslant 1 - cx^{\frac{1}{4}+\frac{1}{2}(g_0+\varepsilon)} \tag{69}$$

for ε and x sufficiently small, where we have defined $g_0 = g(0)$. Hence,

$$\alpha \leqslant \frac{1}{4} + \frac{g_0}{2} = \frac{a}{4}. \tag{70}$$

Similarly we conclude that

$$\bar{Q}(x) \leqslant cx^{-\frac{a}{4}}. \tag{71}$$

If $a \geqslant 2$ the upper bound on α obtained above is replaced by the trivial upper bound $\alpha \leqslant \frac{1}{2}$ coming from the comparison with the comb with no teeth and similarly for the exponent of $\bar{Q}$.

We now turn to the proof of the lower bound. We first note that

$$\bar{Q}(x) \geqslant \frac{1}{1-\bar{P}(x)} \tag{72}$$

by Jensen's inequality. It will suffice to find a suitable lower bound on $\bar{P}(x)$. Noting that the lengths of the teeth are independently and identically distributed we take the expectation of (7) to get

$$\bar{P}(x) = \sum_{\ell=0}^{\infty} \mu_\ell \left\langle \frac{1-x}{3-P_\ell(x)-P_C(x)} \right\rangle. \tag{73}$$

Applying Jensen's inequality again we obtain

$$\bar{P}(x) \geqslant \frac{1-x}{3-\bar{P}_T(x)-\bar{P}(x)}, \tag{74}$$

where

$$\bar{P}_T(x) = \sum_{\ell=0}^{\infty} \mu_\ell P_\ell(x). \tag{75}$$

By rearranging inequality (74) and using the fact that $\bar{P}(x) \leqslant 1$ it follows that

$$\bar{P}(x) \geqslant 1 - \sqrt{1+x-\bar{P}_T(x)}. \tag{76}$$

Using expression (12) for $P_\ell(x)$ and definition (27), we can write

$$1 - P_\ell(x) = \sqrt{x}\frac{1-\mathrm{e}^{-m(x)\ell}}{1+\mathrm{e}^{-m(x)\ell}}, \tag{77}$$

where we have put $m(x) = 2m_\infty(x)$ for convenience. It follows that

$$1 - \bar{P}_T(x) \leqslant \sqrt{x}\sum_{\ell=0}^{\infty} \mu_\ell (1-\mathrm{e}^{-\ell m(x)}). \tag{78}$$

Consider first the case $a > 2$. Then, from (78), we obtain the bound

$$1 - \bar{P}_T(x) \leqslant \sqrt{x}m(x)\sum_{\ell=0}^{\infty} \ell\mu_\ell \leqslant cx. \tag{79}$$

Hence, $\bar{P}(x) \geqslant 1 - c\sqrt{x}$ and we conclude that $\alpha = \frac{1}{2}$ and $d_s = 1$.

More generally define

$$I_\gamma = \sum_{\ell=0}^{\infty} \mu_\ell \ell^\gamma \tag{80}$$

for $\gamma \geqslant 0$. Let

$$\gamma_0 = \sup\{\gamma \geqslant 0 : I_\gamma < \infty\}. \tag{81}$$

For the power law distribution $\mu_\ell = c_a \ell^{-a}$, we see that $\gamma_0 = a - 1$. For $a \leqslant 2$ it follows by a similar argument as before, using

$$1 - \mathrm{e}^{-z} \leqslant z^p, \qquad z \geqslant 0, \qquad 0 \leqslant p \leqslant 1, \tag{82}$$

that for $\varepsilon > 0$,

$$1 - \bar{P}_T(x) \leqslant c\sqrt{x}m(x)^{\gamma_0-\varepsilon}. \tag{83}$$

We conclude from (76) that

$$\bar{P}(x) \geqslant 1 - cx^{\frac{1}{4}(1+\gamma_0-\varepsilon)} \tag{84}$$

so $\alpha \geqslant a/4$. Combining this with (70) shows that $\alpha = a/4$ and therefore the spectral dimension is given by

$$d_s = \frac{4-a}{2} \tag{85}$$

for $1 < a \leqslant 2$.

Let us now consider the general case when μ_ℓ is an arbitrary probability distribution and assume first that γ_0 defined as above is finite. For simplicity, let us further assume[4] that there exists a non-increasing function $g(\varepsilon)$ such that $p(x) \sim x^{g(\varepsilon)}$. Then

$$\lim_{\varepsilon\downarrow 0} g(\varepsilon) = g_0 = \tfrac{1}{2}\gamma_0. \tag{86}$$

In order to prove this let $\gamma > 0$ and consider the integral

$$\begin{aligned} J &= \int_0^1 p(x)x^{-\frac{1}{2}\gamma-1}\mathrm{d}x \\ &= \int_0^1 \sum_{\ell=[x^{-\frac{1}{2}-\varepsilon}]}^{\infty} \mu_\ell x^{-\frac{1}{2}\gamma-1}\mathrm{d}x \\ &\leqslant \sum_{\ell=1}^{\infty} \mu_\ell \int_{(\ell+1)^{-\frac{2}{1+2\varepsilon}}}^{1} x^{-\frac{1}{2}\gamma-1}\mathrm{d}x. \end{aligned} \tag{87}$$

Doing the x integral above we see that

$$J \leqslant -c + \frac{2}{\gamma}\sum_{\ell=1}^{\infty}\mu_\ell(\ell+1)^{\frac{\gamma}{1+2\varepsilon}} \tag{88}$$

which is finite if $\gamma < (1+2\varepsilon)\gamma_0$. Similarly we can show that

$$J \geqslant -c' + \frac{2}{\gamma}\sum_{\ell=1}^{\infty}\mu_\ell \ell^{\frac{\gamma}{1+2\varepsilon}} \tag{89}$$

and the right-hand side in the above inequality diverges if $\gamma > (1+2\varepsilon)\gamma_0$. We conclude that

$$g(\varepsilon) = \tfrac{1}{2}\gamma_0(1+2\varepsilon) \tag{90}$$

so $g_0 = \frac{1}{2}\gamma_0$. This, together with the previous arguments, proves that $d_s = 1$ if $1 < \gamma_0 < \infty$ and $d_s = (3-\gamma_0)/2$ if $0 \leqslant \gamma_0 \leqslant 1$. It is easy to check that $d_s = 1$ for probability distributions which have $\gamma_0 = \infty$.

5. Random trees

In this section, we show how the results and methods of the previous sections can be used to bound the spectral dimension of infinite planar random trees. We begin by recalling some results about such trees from [19]. We consider rooted trees where the root has order 1. There is a measure π on these trees obtained as a limit of the uniform measures on ensembles of trees with a finite number of vertices. With respect to this measure there is with probability 1 a unique infinite simple path in a random tree τ which can be viewed as a 'spine' N_∞ with

[4] This assumption is not true for arbitrary probability distributions and in that case the subsequent arguments will have to be modified. We will leave this technical point aside.

finite trees attached to the spine. For simplicity of presentation, let us consider trees whose vertices have order 1, 2 or 3. Then there is at most a single rooted tree τ_j with root of order 1 attached to each vertex $j \neq 0$ on the spine of the infinite random tree. It is shown in [19] that the trees τ_j are identically and independently distributed with the probability distribution

$$\rho(\{\tau_j\}) = Z^{-1} 3^{-|\tau_j|}, \tag{91}$$

where $|\tau_j|$ denotes the number of links in τ_j and Z is a normalization factor (which in this particular case happens to be equal to 1).

The spectral dimension for the infinite random tree d_s^{tree} is now defined in the same way as the spectral dimension for random combs by considering the return probability to the root for a simple random walk on the random tree. We will show that

$$\tfrac{5}{4} \leqslant d_s^{\text{tree}} \leqslant \tfrac{3}{2}. \tag{92}$$

The lower bound is obtained by showing that the generating function for first return of random walks to the root on the random tree, $P_{\text{tree}}(x)$, is bounded from the above by the corresponding generating function for random combs whose tooth lengths have a power law distribution with exponent $3/2$. The upper bound is obtained by a convexity argument. The result (92) is in agreement with $d_s^{\text{tree}} = 4/3$ found in [6] by different methods.

In order to prove (92), we first note that

$$c_1 n^{-\frac{3}{2}} \leqslant \rho(\{\tau : |\tau| = n\}) \leqslant c_2 n^{-\frac{3}{2}}, \tag{93}$$

see, e.g., [1]. By lemma A and (85), we see that it is sufficient for the lower bound in (92) to show that among all the finite rooted trees τ with a given number ℓ of links, it is N_ℓ which has the largest first return generating function. In order to prove this, consider a tree τ with ℓ links and let $P_\tau(x)$ be its first return generating function. Let v be a vertex in τ at a maximal distance from the root. Let ω be the unique simple path in τ from the root to v. Then v has order 1 and τ can be viewed as a comb with a finite spine ω and finite trees (possibly empty) attached to each vertex of ω. Let v_j, $j = 0, 1, \ldots, n$, be the vertices of ω, $v_n = v$. Let t_j be the tree attached to v_j. Let t_k be the first non-empty tree we encounter as we move from v along ω towards the root. Let τ' be the tree obtained by swapping t_k and t_{k+1} (which is empty by hypothesis). By the argument of the rearrangement lemma we deduce that $P_\tau(x) \leqslant P_{\tau'}(x)$. Taking now a vertex v' in τ' at a maximal distance from the root and repeating the above argument we construct a sequence of trees $\tau^{(i)}$ and for some finite value $i = i_0$ we obtain $\tau^{(i_0)} = N_\ell$.

In order to prove the upper bound in (92), we begin by noting that by Jensen's inequality

$$P_{\text{tree}}(x) \geqslant \frac{1 - x}{3 - P_{\text{tree}}(x) - \bar{P}_t(x)}, \tag{94}$$

where

$$\bar{P}_t(x) = \frac{1}{Z} \sum_{\tau} 3^{-|\tau|} P_\tau(x) \tag{95}$$

and the sum in (95) runs over all finite rooted trees with vertices of order 1, 2 or 3. By the same argument as the one leading to (76) we find that

$$P_{\text{tree}}(x) \geqslant 1 - \sqrt{1 + x - \bar{P}_t(x)}. \tag{96}$$

We will now show that

$$\bar{P}_t(x) \geqslant 1 - c\sqrt{x} \tag{97}$$

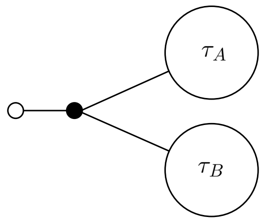

Figure 2. A decomposition of a tree into two subtrees.

which implies the desired result. Given a tree τ it can always be decomposed into the root link and two subtrees τ_A and τ_B as indicated in figure 2. Note that τ_A and/or τ_B may be empty. This allows us to write

$$\bar{P}_t(x) = \frac{1}{3}(1-x) + Z^{-1}\sum_{\tau}\left(\frac{1}{3}\right)^{|\tau|+1}\frac{1-x}{2-P_\tau(x)} + Z^{-2}\sum_{\tau_A,\tau_B}\left(\frac{1}{3}\right)^{|\tau_A|+|\tau_B|+1}\frac{1-x}{3-P_{\tau_A}(x)-P_{\tau_B}(x)}. \tag{98}$$

Using Jensen's inequality again and $Z = 1$ we obtain

$$\bar{P}_t(x) \geqslant \frac{1}{3}(1-x)\left(1+\frac{1}{2-\bar{P}_t(x)}+\frac{1}{3-2\bar{P}_t(x)}\right). \tag{99}$$

Rearranging (99) leads to

$$(6\bar{P}_t(x)-11)(\bar{P}_t(x)-1)^2 + x\big(2\bar{P}_t^2(x)-10\bar{P}_t(x)+11\big) \geqslant 0 \tag{100}$$

which implies (97) since $\bar{P}_t(x) \to 1$ as $x \to 0$.

6. Some non-random combs

In this section, we use the technique s we have developed so far to calculate the spectral dimension of some non-random combs. We first remark that it follows easily from the previous results that any periodic comb has either spectral dimension 3/2 (if it has some infinite teeth) or spectral dimension 1 if all teeth have finite length.

We will discuss two different types of combs: (i) combs with infinite teeth where the distance between tooth n and tooth $n+1$ is an increasing function of n and (ii) combs with finite teeth such that the length of tooth n is an increasing function of n. For simplicity of presentation we will choose the increasing functions to be powers but our methods apply to more general cases. In both cases we find that the spectral dimension varies continuously with the power.

6.1. Combs with increasing tooth spacings

Let $a > 0$ and define C to be the comb all of whose teeth are infinite such that the distance from the kth tooth to the $(k+1)$th tooth is

$$L_k = [(k+1)^a], \tag{101}$$

where $[r]$ denotes the integer part of r as before. The distance from the root to the first tooth is $L_0 = 1$. We will show that this comb has spectral dimension

$$d_s = \frac{3+a}{2+a}. \tag{102}$$

We first prove an upper bound on $P_C(x)$. This will be done by an inductive argument bounding the two-point function at a suitable distance along the spine:

$$G_C(x, [x^{-\bar{\eta}-\varepsilon}]) = E(x) \tag{103}$$

for arbitrary $\varepsilon > 0$, where

$$\bar{\eta} = \frac{a+1}{2(a+2)}. \tag{104}$$

For this purpose we need to introduce slightly more general combs than C. For any $f \geqslant 1$ let C^f be the comb with all teeth infinite such that the distance from the kth tooth to the $(k+1)$ th tooth is

$$L_k = [(k+f)^a]. \tag{105}$$

Note that $L_0 = [f^a]$ in this case and $C^1 = C$. The inductive hypothesis is as follows. There exists $\eta_0 \in \left(\bar{\eta}, \frac{1}{2}\right]$ such that for any $\eta > \eta_0$ and any constant $c > 0$ there is a fixed E-function, as defined in (46), and the inequality

$$G_{C^f}(x; [x^{-\eta}]) \leqslant E(x) \tag{106}$$

holds for all f in the range $1 \leqslant f \leqslant cx^{-\frac{\eta}{1+a}}$. By (49) the hypothesis is true for $\eta_0 = \frac{1}{2}$ since (106) holds in this case for any value of $f \geqslant 1$. We shall prove that the statement then holds also with η_0 replaced by $\phi(\eta_0)$, where

$$\phi(\eta) = \frac{1}{4} + \frac{a\eta}{2(a+1)}. \tag{107}$$

The strategy of the following argument is to use the induction hypothesis (106) to obtain an upper bound on $P_{C^f}(x)$ which in turn will be used to prove an improved upper bound on G_{C^f} using the representation (24).

Consider one of the combs C^f. The distance from the root to the nth tooth is given by

$$D_n = \sum_{k=0}^{n-1} [(k+f)^a] \tag{108}$$

and fulfils

$$\frac{1}{a+1}((n+f-1)^{a+1} - (f-1)^{a+1}) \leqslant D_n \leqslant \frac{1}{a+1}((n+f)^{a+1} - f^{a+1}). \tag{109}$$

For a fixed $\eta > \eta_0$ choose n as a function of $x > 0$, such that $D_{n+1} \geqslant [x^{-\eta}] \geqslant D_n$. It follows that

$$L_n^{-1} \sim x^{\frac{\eta a}{a+1}} \tag{110}$$

since $f^{a+1} \leqslant c^{a+1} x^{-\eta}$ by assumption. The contribution to $P_{C^f}(x)$ coming from walks that go beyond D_{n+1} is bounded by $E(x)$ as a consequence of (106). By the rearrangement lemma and (51) it follows that

$$\begin{aligned} P_{C^f}(x) &\leqslant P_{*L_n}(x) + E(x) \\ &\leqslant 1 - cx^{\eta'}, \end{aligned} \tag{111}$$

where $\eta' = \phi(\eta)$ and we have used (110) and (B.2) to obtain the second inequality. In order to complete the inductive step we now need to show that (106) holds with η replaced by η'. So let C^f be given with $1 \leqslant f \leqslant cx^{-\eta'}$ and let $D = [x^{-\eta'}]$. By (24) and the rearrangement lemma we have

$$\begin{aligned} G_{C^f}(x, D) &= \sigma(D)(1-x)^{-\frac{1}{2}D} \prod_{k=0}^{D-1} P_{C_k^f}(x) \\ &\leqslant (1-x)^{-\frac{1}{2}D} (P_{C^{f'}}(x))^D, \end{aligned} \tag{112}$$

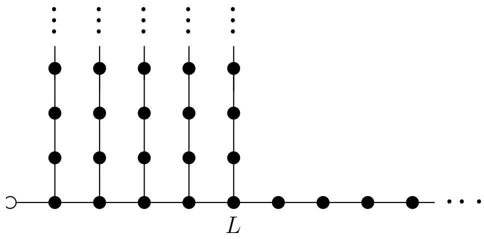

Figure 3. The comb $C(L)$ with $L = 5$.

where the comb $C^{f'}$ is defined such that the distance from its root to the first tooth is larger than or equal to the largest tooth separation up to a distance D along the spine in C^f. For this purpose it suffices to take

$$f' = c' x^{-\frac{\eta'}{1+a}} \tag{113}$$

for a suitable constant c'. The comb $C^{f'}$ is in the class covered by the induction hypothesis with η replaced by $\eta - \varepsilon$ for a suitable ε since $\eta' < \eta$. Hence,

$$P_{C^{f'}}(x) \leqslant 1 - c_1 x^{\phi(\eta-\varepsilon)}, \tag{114}$$

and we conclude from (112) that

$$G_{C^f}(x, [x^{-\eta'}]) = E(x) \tag{115}$$

for $1 \leqslant f \leqslant cx^{-\eta'}$. We have thus proven that if the induction hypothesis holds for a particular η_0 it also holds for $\eta_1 = \phi(\eta_0)$. Defining η_r inductively by

$$\eta_{r+1} = \phi(\eta_r) \tag{116}$$

we see that $\{\eta_r\}$ is a decreasing sequence in the interval $\left[\frac{a+1}{2(a+2)}, \frac{1}{2}\right]$ and the induction hypothesis holds for all η which satisfy

$$\eta > \bar{\eta} = \lim_{r\to\infty} \eta_r = \frac{a+1}{2(a+2)}. \tag{117}$$

We conclude that

$$P_C(x) \leqslant 1 - cx^{\frac{1}{4} + \frac{a\eta}{2(a+1)}} + E(x) \tag{118}$$

for any $\eta > \bar{\eta}$ and therefore

$$P_C(x) \leqslant 1 - cx^{\frac{a+1}{2(a+2)} + \varepsilon} \tag{119}$$

for any $\varepsilon > 0$. This proves that

$$\alpha \leqslant \frac{a+1}{2(a+2)}. \tag{120}$$

In order to prove the corresponding lower bound on $P_C(x)$ let $\eta \in \left(\bar{\eta}, \frac{1+a}{4}\right)$. Let $D_n \sim [x^{-\eta}]$ as before and let C' be the comb C with all teeth beyond the nth one removed. Then from (51) and (106) it follows that

$$P_{C'}(x) = P_C(x) + E(x). \tag{121}$$

Define $C(n)$ to be the comb with infinite teeth at the vertices $1, 2, \ldots, n$ but no other teeth, see figure 3. Then, by the rearrangement lemma and (121),

$$P_C(x) \geqslant P_{C(n)}(x) + E(x). \tag{122}$$

The asymptotic behaviour of $P_{C(n)}(x)$ as $x \to 0$ is computed in appendix A and we find

$$P_C(x) \geqslant 1 - cx^{\frac{1}{2}-\frac{\eta}{a+1}} \tag{123}$$

which implies the desired converse to inequality (119),

$$P_C(x) \geqslant 1 - cx^{\bar{\eta}-\varepsilon} \tag{124}$$

for any $\varepsilon > 0$. We conclude that

$$\alpha = \frac{a+1}{2(a+2)} \tag{125}$$

and the spectral dimension (102) follows.

6.2. *Combs with increasing tooth length*

In this subsection, we consider the comb C for which the length of T_k is given by $\ell_k = [k^a], a > 0$. We will prove that the spectral dimension of this comb is $\frac{3}{2}$ if $a \geqslant 2$ but

$$d_{\mathrm{s}} = \frac{2(1+a)}{2+a} \tag{126}$$

for $0 < a < 2$. In order to prove this we first consider the case $0 < a < 2$ and define the comb C' by

$$\begin{aligned} \ell_k &= 0, \qquad && k < k_0 \\ \ell_k &= \left[k_0^a\right], \qquad && k \geqslant k_0, \end{aligned} \tag{127}$$

where k_0 will be chosen to depend on x below. The comb C' can be obtained from C by shortening teeth so by the monotonicity lemma

$$P_C(x) \leqslant P_{C'}(x). \tag{128}$$

We then set $k_0 = [x^{-\beta}]$, $0 < \beta < \frac{1}{2}$, and choose β to optimize the bound (128). The asymptotic behaviour of the function $P_{C'}(x)$ is computed in appendix A where this function is denoted as $P_{k,\ell}(x)$ with $k = k_0$ and $\ell = \left[k_0^a\right]$.

Taking first the case $\beta a < \frac{1}{2}$ we find

$$P_{C'}(x) = 1 - cx^{\delta} + O(x^{\delta+\varepsilon}), \tag{129}$$

where $\delta = \max\left\{\frac{1-\beta a}{2}, \beta\right\}$ and $\varepsilon > 0$. The value of δ is minimized by choosing $\beta = \beta_{\mathrm{opt}} \equiv \frac{1}{2+a}$ and we conclude that

$$\alpha \leqslant \frac{1}{2+a}, \tag{130}$$

provided that $\beta_{\mathrm{opt}} a < \frac{1}{2}$; that is for $a < 2$.

To obtain a lower bound on $P_C(x)$ we first note that by monotonicity $P_{C_k}(x) \leqslant P_C(x)$, as C_k can be obtained from C by lengthening teeth, and therefore

$$\begin{aligned} G_C(x; n) &= (1-x)^{-n/2}\sigma(n)\prod_{k=0}^{n-1} P_{C_k}(x) \\ &\leqslant 3(1-x)^{-n/2}(P_C(x))^n \\ &\leqslant c_1 \exp\left(-c_2 n x^{\frac{1}{2+a}}\right). \end{aligned} \tag{131}$$

Combining this fact with (51) and the monotonicity lemma shows that

$$P_C(x) \geqslant P_{C''}(x) + E(x), \tag{132}$$

where C'' is the comb with teeth of constant length $\ell_k = \left[k_1^a\right]$ and $k_1 = \left[x^{-\frac{1}{2+a}-\varepsilon}\right]$. This comb has exactly the structure of the comb # considered in appendix A. We find, using (A.30),

$$P_{C''}(x) \geqslant 1 - cx^{\frac{1}{2+a}-\varepsilon'} \tag{133}$$

for any $\varepsilon' > 0$. It follows that $\alpha \geqslant \frac{1}{2+a}$; combining this with (130) gives the results

$$\alpha = \frac{1}{2+a}, \qquad d_s = \frac{2+2a}{2+a}, \qquad 0 < a < 2. \tag{134}$$

We now turn to the case $a \geqslant 2$. We use the argument leading to (129) with $\beta = \frac{1}{4}$. The teeth of C' are now so long that they are effectively infinite and (A.18), (A.25) and (A.26) yield

$$P_C(x) \leqslant P_{C'}(x) = 1 - cx^{\frac{1}{4}} + O\left(x^{\frac{1}{4}+\varepsilon}\right), \tag{135}$$

where $\varepsilon > 0$. From (20) we know that $P_C(x) \geqslant P_*(x)$ and it follows immediately that

$$\alpha = \tfrac{1}{4}, \qquad d_s = \tfrac{3}{2}, \qquad a \geqslant 2. \tag{136}$$

7. Anomalous diffusion

In this section, we explore the connection between the full heat kernel on random combs and the functions we have focused on so far, namely, the two-point function and the first return generating function. The main result is that anomalous diffusion along the spine is described by the decay of the two-point function and the critical exponents α and ν coincide. We will focus on the random comb with random tooth length. The comb with random spacing between infinitely long teeth can be treated by similar arguments.

7.1. The exponents α and ν are equal

Our starting point is the representation (24) of the two-point function on a comb C. We will bound ν from below with α using a convexity argument. The opposite inequality follows from pointwise estimates as obtained in sections 3 and 4 in the calculation of the spectral dimension.

Using (7) and (24), and remarking that

$$\frac{1}{1+y} \geqslant \mathrm{e}^{-y} \tag{137}$$

for any $y \geqslant 0$, we find

$$G_C(x;n) \geqslant \sigma(n)(1-x)^{n/2} \exp\left(-\sum_{k=0}^{n-1}\left(2 - P_{T_{k+1}}(x) - P_{C_{k+1}}(x)\right)\right). \tag{138}$$

Averaging over the comb ensemble and applying Jensen's inequality we obtain

$$\bar{G}(x;n) \geqslant (1-x)^{n/2}\,\mathrm{e}^{-n(2-\bar{P}_T(x)-\bar{P}(x))}, \tag{139}$$

where $\bar{P}_T(x)$ is the average of the first return generating functions on the individual teeth. Clearly, $\bar{P}_T(x) \geqslant \bar{P}(x)$ so

$$\bar{G}(x;n) \geqslant (1-x)^{n/2}\,\mathrm{e}^{-2n(1-\bar{P}(x))}. \tag{140}$$

We conclude immediately that $\nu \geqslant \alpha$.

In order to prove the converse inequality it is sufficient to show that

$$\bar{G}(x;[x^{-\alpha-\varepsilon'}]) = E(x) \tag{141}$$

for arbitrarily small $\varepsilon' > 0$. Indeed, it follows from the definition of the mass and (40) that

$$\bar{G}(x; n) \geqslant \bar{G}^0(x; n) \geqslant \mathrm{e}^{-m(x)n} \tag{142}$$

for all n. Hence (141) implies $m(x)x^{-\alpha-\varepsilon'} \to \infty$ as $x \to 0$, which shows that $\nu \leqslant \alpha + \varepsilon'$.

To establish (141), we split the average over $\mathcal{C}$ into a contribution from $\mathcal{B}_\varepsilon$ and a contribution from $\mathcal{C}\backslash\mathcal{B}_\varepsilon$ as we did in section 4 for $\bar{P}(x)$. By (28) the former contribution is bounded from the above by

$$\tfrac{3}{2}G_\infty(x; n)\pi(\mathcal{B}_\varepsilon) = E(x). \tag{143}$$

For $C \in \mathcal{C}\backslash\mathcal{B}_\varepsilon$ we recall from section 4 that the first return generating function can be estimated by

$$P_C(x) \leqslant P_{*M}(x) + E(x), \tag{144}$$

where $M = [x^{-g(\varepsilon)-\varepsilon}]$, and $E(x)$ is independent of C. We claim that the bound (144) also holds for C_k, $k = 1, \ldots, [x^{-\alpha-\varepsilon'}]$, uniformly in $\mathcal{C}\backslash\mathcal{B}_\varepsilon$ for ε' sufficiently small. Recalling that combs $C \in \mathcal{C}\backslash\mathcal{B}_\varepsilon$ are characterized by the requirement $L_0, \ldots, L_{K-1} \leqslant M$, where $K = \left[x^{-\frac{1}{2}-\varepsilon}\right]$, and that $\frac{1}{4} \leqslant \alpha \leqslant \frac{1}{2}$, we choose ε' such that $\alpha + \varepsilon' < \frac{1}{2} + \varepsilon$, and hence $x^{-\alpha-\varepsilon'}/K \to 0$ as $x \to 0$. In particular, for each C_k under consideration, we have that $L_0, \ldots, L_{[\frac{1}{2}K]} \leqslant M$, for x sufficiently small, and inspection of the proof of (144) shows that the inequality holds for the C_ks as claimed (with a modified E-function). We can thus use (144) together with the product representation (24) and obtain for the second contribution to $\bar{G}(x; [x^{-\alpha-\varepsilon'}])$ the bound

$$(1 - x)^{n/2}(P_{*M}(x) + E(x))^{[x^{-\alpha-\varepsilon'}]} = E(x). \tag{145}$$

We have thus proved (141) and hence also $\nu \leqslant \alpha$.

7.2. The heat kernel

Consider a comb C and define $K_C(t; n, k)$ as the probability that a random walker who leaves the root at time 0 is located at the vertex k in the nth tooth of C at time t. We will denote this vertex as (n, k). If the nth tooth of C has length smaller than k we define this probability to be 0. We will refer to the function $K_C(t; n, k)$ as the heat kernel on C since it satisfies the heat (or diffusion) equation on C:

$$K(t + 1; n, k) = \sum_{(n',k')} \sigma(n', k')^{-1} K(t; n', k'), \tag{146}$$

where the sum in (146) runs over the nearest neighbours of the vertex (n, k) in C. Next define the function

$$K_C(t; n) = \sum_{k=0}^{\infty} K_C(t; n, k) \tag{147}$$

which is the probability that a walker has travelled a distance n along the spine at time t. We are interested in the asymptotic behaviour of $K_C(t; n)$ for large t and n. In order to analyse this function we define the corresponding Green function by the Laplace transformation

$$H_C(x; n) = \sum_{t=0}^{\infty} (1 - x)^{t/2} K_C(t; n). \tag{148}$$

Decomposing the walks contributing to the heat kernel we can express $H_C(x; n)$ in terms of previously defined generating functions as

$$H_C(x; n) = \frac{G_C(x; n)}{1 - P_C(x)} D_\ell(x), \tag{149}$$

where ℓ is the length of the nth tooth of C and

$$D_\ell(x) = 1 + \frac{1}{3}\sum_{k=1}^{\ell} G_{N_\ell}(x;k). \tag{150}$$

Using (24) and (12) we can write $D_\ell(x)$ in terms of $P_\ell(x)$ as

$$D_\ell(x) = \frac{2}{3} + \frac{1}{3}(1+\sqrt{1-x})\left(\frac{1-P_\ell(x)}{x}\right). \tag{151}$$

We begin by establishing a lower bound on H_C. Let $C(\infty n)$ denote the comb C with the nth tooth replaced by an infinite tooth. By the monotonicity lemma and (24) it follows that

$$P_{C(\infty n)}(x) \leqslant P_C(x), \qquad G_{C(\infty n)}(x;n) \leqslant \tfrac{3}{2} G_C(x;n). \tag{152}$$

Hence,

$$H_C(x;n) \geqslant \frac{(1-x)^{-n/2} D_\ell(x)}{1-P_{C(\infty n)}(x)} \prod_{k=0}^{n-1} P_{C_k(\infty n-k)}(x). \tag{153}$$

We note that $D_\ell(x)$ is the only quantity on the right-hand side of inequality (153) which depends on the length of the nth tooth of C. Now consider one of the combs $C_k(\infty n-k)$ and swap teeth $(n-k-1)$ times so that the infinite tooth becomes the first tooth while the tooth T_j of $C_k(\infty n-k)$ becomes tooth number $j+1$ for $j = 1, 2, \ldots, n-k-1$. Denote the resulting comb by $C'_k(\infty n-k)$. By the rearrangement lemma $P_{C_k(\infty n-k)}(x) \geqslant P_{C'_k(\infty n-k)}(x)$. By (7),

$$P_{C'_k(\infty n-k)}(x) = \frac{1-x}{3-P_\infty(x)-P_{\tilde{C}_{k+1}}(x)}, \tag{154}$$

where $\tilde{C} = C'(\infty n)$. We now average over C. Using (137), combined with (154), and Jensen's inequality we obtain,

$$\langle H_C(x;n)\rangle \geqslant \frac{(1-x)^{n/2}\langle D_\ell(x)\rangle}{1-\langle P_{C(\infty n)}(x)\rangle} \mathrm{e}^{-n(2-P_\infty(x)-\bar{P}(x))}. \tag{155}$$

We now take the ensemble average of D_ℓ. Let us first assume that the teeth are finitely long with probability 1 and let γ_0 be given by (81). Then, by (83),

$$1-\langle P_\ell(x)\rangle \leqslant c x^{\frac{1}{2}(1+\gamma_0-\varepsilon)}. \tag{156}$$

The converse inequality, with ε replaced by $-\varepsilon$, follows from (69) and (84) so

$$1-\langle P_\ell(x)\rangle \sim x^{2\alpha}. \tag{157}$$

We conclude from (151) that

$$\langle D_\ell(x)\rangle \geqslant c_1 + c_2 x^{2\alpha-1}. \tag{158}$$

By (152), $\langle P_{C(\infty n)}(x)\rangle \leqslant \bar{P}(x)$. On the other hand, (154) with $k=0$ and Jensen's inequality imply that

$$\langle P_{C(\infty n)}(x)\rangle \geqslant \frac{1-x}{3-P_\infty(x)-\bar{P}(x)} \tag{159}$$

and we conclude that

$$1-\bar{P}(x) \sim 1-\langle P_{C(\infty n)}(x)\rangle. \tag{160}$$

Combining (155), (158) and (160) yields

$$\langle H_C(x;n)\rangle \geqslant c_1 x^{\alpha-1}\,\mathrm{e}^{-c_2 n x^{\alpha}}. \tag{161}$$

In order to establish the corresponding upper bound we use inequalities (144) and (145) which imply that

$$H_C(x;n) \leqslant \frac{(1-x)^{-n/2}D_\ell(x)}{1-P_{*M}(x)}((P_{*M}(x)+E_1(x))^n + E_2(x)(P_\infty(x))^n), \qquad (162)$$

where the E-functions E_1 and E_2 are uniform in C. It is clear from (151) that the converse of inequality (158) holds (with different constants c_1 and c_2). Hence,

$$\langle H_C(x;n)\rangle \sim x^{\alpha-1}\,\mathrm{e}^{-c_2 n x^\alpha} + E(x)\,\mathrm{e}^{-n\sqrt{x}} \qquad (163)$$

for small x. If infinite teeth appear with non-zero probability in the comb ensemble, then $\alpha = \frac{1}{2}$ and we can ignore the finite teeth since they do not affect the critical behaviour. In this case, (163) holds with $\alpha = \frac{1}{2}$.

The asymptotic relation (163) enables us to compute the mean extent down the spine of walks at large time. We have, for $k > 0$,

$$\begin{aligned}\sum\nolimits_t (1-x)^{t/2}\langle\langle n^k\rangle_{\omega:|\omega|=t}\rangle_C &= \sum_n n^k\langle H_C(x;n)\rangle \\ &\sim \frac{c_1}{x^{1+k\alpha}} + E(x), \qquad x\to 0,\end{aligned} \qquad (164)$$

and so, by standard Tauberian theorems (see, e.g., [23, 24]),

$$\langle\langle n^k\rangle_{\omega:|\omega|=t}\rangle_C \sim c_2 t^{k\alpha}, \qquad t\to\infty. \qquad (165)$$

On the other hand, (163) does of course not allow us to compute $\langle K_C(t;n)\rangle$. However, if we make the ansatz,

$$\langle K_C(t;n)\rangle \approx c\frac{n^\delta}{t^\gamma}\,\mathrm{e}^{-c' n^\beta/t^\epsilon}, \qquad (166)$$

we find that this is consistent with (163) if and only if

$$\beta = \frac{1}{1-\alpha}, \qquad \epsilon = \frac{\alpha}{1-\alpha}, \qquad \gamma = \frac{\alpha}{2(1-\alpha)}, \qquad \delta = \frac{2\alpha-1}{2(1-\alpha)}. \qquad (167)$$

This is in agreement with the exact results for the half line [20] and the full comb [18]. We believe that (166) captures the essential behaviour of the averaged heat kernel for

$$\frac{n}{t^\alpha} \gg 1. \qquad (168)$$

Of course, $K_C(t;n) = 0$ for $t < n$ and for n slightly smaller than t it is clear that $K_C(t;n)$ decays exponentially in n. We also know that $\langle K_C(t;0)\rangle \sim t^{-d_s/2}$. We believe that $\langle K_C(t;n)\rangle$ is a decreasing function of n for a fixed t but a proof does not seem to be straightforward.

7.3. First passage

Let $q_C(t;n)$ be the probability that a walk which leaves the root at time $t=0$ on a comb C hits the vertex n on the spine for the first time at time t. We define the corresponding generating function as

$$U_C(x;n) = \sum_{t=0}^\infty (1-x)^{t/2} q(t;n). \qquad (169)$$

The mean first passage time at n is defined as the quantity

$$\bar{t}_C(n) = \sum_{t=0}^\infty t q_C(t;n) = -2\,\frac{\partial}{\partial x}U_C(x;n)\bigg|_{x=0}. \qquad (170)$$

One can calculate the probability $q_{N_\infty}(t;n)$ explicitly by elementary combinatorics with the result

$$U_{N_\infty}(x;n) = \frac{1}{\cosh(m_\infty(x)n)} \tag{171}$$

which implies that the mean first passage time on the half line is $\bar{t}_\infty(n) = n^2$.

In this paper, we shall not attempt to make a full calculation of the averaged probability distribution $\langle q_C(t;n)\rangle$ but rather be content with showing that all the average mean first passage times are infinite on random combs with $\alpha < \frac{1}{2}$. For this purpose, it clearly suffices to show that $\langle \bar{t}_C(2)\rangle$ is infinite. The physical reason for this is easily seen to be that if the teeth in a random comb are sufficiently long to shift the spectral dimension away from 1 then the average time spent in a random tooth is infinite.

Let us first assume that we have a random comb where there is a non-vanishing probability p that we have an infinite tooth at each vertex on the spine. Then

$$\langle \bar{t}_C(2)\rangle \geqslant -\left(\frac{2p}{9}\right)\frac{\mathrm{d}}{\mathrm{d}x}P_\infty(x)\bigg|_{x=0} = \infty, \tag{172}$$

where the lower bound is obtained by taking into account only combs with an infinite first tooth and restricting the attention to walks that wander into the first tooth and proceed directly to the vertex 2 when they return from the first tooth. Similarly, if the teeth are finite with probability distribution μ_ℓ,

$$\langle \bar{t}_C(2)\rangle \geqslant -\frac{2}{9}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{\ell=1}^{\infty}\mu_\ell P_\ell(x)\bigg|_{x=0}. \tag{173}$$

Using (12) the right-hand side in (173) is easily seen to be bounded from below by

$$c\lim_{x\to 0}\sum_{\ell=0}^{\infty}\mu_\ell \ell\, \mathrm{e}^{-m_\infty(x)\ell} \tag{174}$$

which is infinite if $\alpha = (\gamma_0+1)/4 < 1/2$ where γ_0 is defined by (81).

8. Discussion

Heat kernels on graphs and Riemannian manifolds have been extensively studied by mathematicians, see, e.g., [25, 26] and references therein. Much of this work is aimed at establishing the connection between pointwise behaviour of the heat kernel and geometrical properties of the graphs and manifolds. The most relevant results from our point of view are inequalities which, in our notation, are written as

$$\frac{2d_{\mathrm{H}}}{1+d_{\mathrm{H}}} \leqslant d_{\mathrm{s}} \leqslant d_{\mathrm{H}}, \tag{175}$$

valid for graphs where the Hausdorff dimension exists and is finite, see [25] theorems 2.2 and 2.3. The Hausdorff and spectral dimensions calculated in this paper are readily seen to satisfy these bounds, some saturate the lower bound, others saturate the upper bound and some are in between, see table 1.

It is not understood in detail which properties of a graph cause the spectral and Hausdorff dimensions to differ. For the special and rather simple example of combs it is easy to see that if we have mostly short teeth, then the spectral and Hausdorff dimensions are both equal to 1. As the teeth grow, both dimensions grow but the Hausdorff dimension grows faster in general. It would be interesting to relate the two dimensions to the distribution of the orders of vertices.

Table 1. The spectral and Hausdorff dimensions discussed in this paper.

	d_{H}	d_{s}	$\frac{2d_{\mathrm{H}}}{1+d_{\mathrm{H}}}$
Random tooth spacing	2	$\frac{3}{2}$	$\frac{4}{3}$
Random tooth length $a \geqslant 2$	1	1	1
Random tooth length $1 < a < 2$	$3 - a$	$\frac{4-a}{2}$	$\frac{6-2a}{4-a}$
Growing tooth spacing	$\frac{2+a}{1+a}$	$\frac{3+a}{2+a}$	$\frac{4+2a}{3+2a}$
Growing tooth length $a \geqslant 2$	2	$\frac{3}{2}$	$\frac{4}{3}$
Growing tooth length $1 \leqslant a < 2$	2	$\frac{2(1+a)}{2+a}$	$\frac{4}{3}$
Growing tooth length $0 < a < 1$	$1 + a$	$\frac{2(1+a)}{2+a}$	$\frac{2+2a}{2+a}$
Random trees	2	$\frac{4}{3}$	$\frac{4}{3}$

For more general graphs the connectivity clearly plays a role, not only the order distribution.

There is no analytical understanding of the spectral dimension of random surfaces and higher dimensional random manifolds. The spectral dimension of random surfaces is believed to be 2 [5, 10] while the Hausdorff dimension is known to be 4 [27], see also [28, 29]. It has been shown recently [28, 29] that the generic structure of infinite planar random surfaces (triangulations) is analogous to that of the random infinite trees discussed in section 5. If we take such a surface S with a marked vertex and look at the boundary of a ball B, the boundary will have a number of disjoint components and with probability 1 only one of these components bounds an infinite subsurface of S. This means that we can view the infinite planar random surface as a tube with finite size outgrowths (baby universes) which are in fact distributed in a simple way analogous to (91). The tube and the outgrowths on the random surface correspond to the spine and the teeth of the random comb. Whether this picture allows us to obtain a rigorous control over the spectral dimension of random surfaces remains to be seen.

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Appendix A

In this appendix, we compute the first return generating functions for a number of simple combs. The method is essentially to solve the recursion relation (7) in simple cases.

We begin by evaluating the first return generating function $P_\ell(x)$ for finite teeth. We note that $P_1(x) = 1 - x$ and

$$P_\ell(x) = \frac{1-x}{2 - P_{\ell-1}(x)}. \tag{A.1}$$

Let us define

$$\Delta_\ell = \frac{P_\ell - P_\infty}{2 - P_\infty}. \tag{A.2}$$

It follows from (A.1) that Δ_ℓ satisfies the recursion relation

$$\Delta_\ell = \left(\frac{P_\infty}{2-P_\infty}\right)\frac{\Delta_{\ell-1}}{1-\Delta_{\ell-1}}. \tag{A.3}$$

Writing

$$A = \frac{2-P_\infty}{P_\infty} \qquad \text{and} \qquad X_\ell = \Delta_\ell^{-1} \tag{A.4}$$

we see that (A.3) can be written as

$$X_\ell = A(X_{\ell-1}-1) \tag{A.5}$$

which has the solution

$$X_{\ell+1} = A^\ell X_1 - A\frac{1-A^\ell}{1-A}. \tag{A.6}$$

Inserting the values of A and X_1 and doing some algebra lead to the desired result (12).

Next, let us consider the comb $C(L)$ which has infinite teeth at $1, 2, \ldots, L$ but no other teeth. Then, by (7),

$$P_{C(L)}(x) = \frac{1-x}{3-P_\infty-P_{C(L-1)}(x)}. \tag{A.7}$$

Defining

$$E_L = \frac{P_{C(L)}-P_*}{3-P_\infty-P_*} \tag{A.8}$$

we find from (A.7) that E_L satisfies the recursion relation

$$E_L = \frac{P_*}{3-P_\infty-P_*}\frac{E_{L-1}}{1-E_{L-1}} \tag{A.9}$$

which is of the same form as (A.3). Hence, by the same reasoning as that leading to (A.6) we find that

$$E_L = \frac{1}{B^L E_0^{-1} - B\frac{1-B^L}{1-B}}, \tag{A.10}$$

where

$$B = \frac{3-P_\infty-P_*}{P_*}. \tag{A.11}$$

Hence,

$$P_{C(L)} - P_* = \frac{P_\infty - P_*}{1+F(B^L-1)}, \tag{A.12}$$

with

$$F = 1 + \frac{P_\infty-P_*}{P_*(B-1)}. \tag{A.13}$$

Noting that $B = 1 + 2x^{1/4} + O(\sqrt{x})$ as $x \to 0$ we see that

$$F = \tfrac{3}{2} + O(x^{1/4}). \tag{A.14}$$

The comb $C(L)$ clearly has spectral dimension 1 since it only has a finite number of teeth. However, for our application in section 6, we are interested in the behaviour of $P_{C(L)}(x)$ as $x \to 0$ with L behaving like a negative power of x.

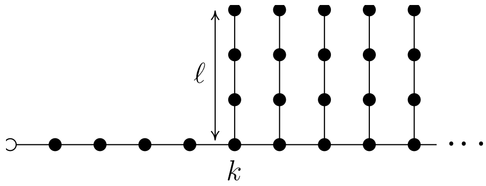

Figure 4. The comb $C(k, \ell)$ with $k = 5$ and $\ell = 4$.

Let us now assume that $L = [x^{-\beta}]$ with $0 < \beta < \frac{1}{4}$. It follows by expanding out the denominator in (A.12) that

$$P_{C(L)}(x) = 1 - 3x^{\frac{1}{2}-\beta} + o\big(x^{\frac{1}{2}-\beta}\big). \tag{A.15}$$

We conclude that

$$1 - P_{C(L)}(x) \sim x^{\frac{1}{2}-\beta} \tag{A.16}$$

which was needed for inequality (123).

The final comb we consider in this section has no teeth at vertices $0, 1, \ldots, k-1$ but teeth of length ℓ at $k, k+1, \ldots$. Denote this comb by $C(k, \ell)$, see figure 4. We will be interested in the limit of the first return generating function on $C(k, \ell)$, denoted as $P_{k,\ell}$, when k and ℓ tend to infinity as $x \to 0$. The logic of the calculation is very much the same as above.

Let us denote $P_{1,\ell}$ by $P_\#$ and note that this function satisfies the recursion relation

$$P_\#(x) = \frac{1-x}{3 - P_\ell(x) - P_\#(x)} \tag{A.17}$$

which is easily solved with the result

$$P_\#(x) = \frac{3 - P_\ell(x)}{2} - \sqrt{1 - P_\ell(x) + \frac{1}{4}(1 - P_\ell(x))^2 + x}. \tag{A.18}$$

With A defined as in (A.4) one finds

$$\frac{2 - P_\infty}{P_{k,\ell} - P_\infty} = A^{k-1}\frac{2 - P_\infty}{P_\# - P_\infty} - A\frac{1 - A^{k-1}}{1 - A}, \tag{A.19}$$

cf (A.6). This can be written as

$$P_{k,\ell} - P_\infty = \frac{2(P_\# - P_\infty)(1 - P_\infty)}{2(1 - P_\infty) + (2 - P_\infty - P_\#)(A^{k-1} - 1)}. \tag{A.20}$$

We now wish to find the asymptotic behaviour of $P_{k,l}(x)$ as $x \to 0$ with $k = [x^{-\beta}]$ and $\ell = [k^a]$. Assume that $0 < \beta < \frac{1}{2}$ and recall that

$$A = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}. \tag{A.21}$$

It follows that

$$A^k - 1 = 2x^{\frac{1}{2}-\beta} + o\big(x^{\frac{1}{2}-\beta}\big). \tag{A.22}$$

Note that (12) can be written as

$$1 - P_\ell(x) = \sqrt{x}\frac{A^\ell - 1}{A^\ell + 1}, \tag{A.23}$$

so if $\beta a < \frac{1}{2}$ and we use (A.22) with β replaced by βa we find

$$1 - P_\ell(x) = x^{1-\beta a} + o(x^{1-\beta a}). \tag{A.24}$$

In the case $\beta a > \frac{1}{2}$ it is not hard to see that

$$1 - P_\ell(x) = \sqrt{x} + E(x). \tag{A.25}$$

In the limiting case $\beta a = \frac{1}{2}$ an easy calculation yields

$$1 - P_\ell(x) = \frac{\mathrm{e}^2 - 1}{\mathrm{e}^2 + 1}\sqrt{x} + o(\sqrt{x}). \tag{A.26}$$

The next step is to find how $P_\#(x)$ behaves as $x \to 0$ and then we can infer the behaviour of $P_{k,\ell}(x)$ from (A.20). It is convenient to split the argument into two cases.

Case 1. $\beta a \geqslant \frac{1}{2}$

It is clear from (A.18) that $P_\#(x) = 1 - cx^{\frac{1}{4}} + o\big(x^{\frac{1}{4}}\big)$. Using this we find that

$$P_{k,\ell}(x) - P_\infty(x) = -\frac{cx^{\frac{1}{4}+\beta} + o\big(x^{\frac{1}{4}+\beta}\big)}{x^\beta + cx^{\frac{1}{4}} + o\big(x^{\frac{1}{4}}\big)}. \tag{A.27}$$

If $\beta < \frac{1}{4}$, then we find

$$P_{k,\ell}(x) = P_\infty(x) - cx^{\frac{1}{4}} + o\big(x^{\frac{1}{4}}\big) \tag{A.28}$$

and if $\beta > \frac{1}{4}$, then

$$P_{k,\ell}(x) = P_\infty(x) - x^\beta + o(x^\beta). \tag{A.29}$$

In the crossover case $\beta = \frac{1}{4}$ we find (A.28) with the constant c replaced by $c(1+c)^{-1}$. We remark that this calculation is insensitive to the value of a.

Case 2. $\beta a < \frac{1}{2}$

Using (A.18) and (A.24) we obtain

$$P_\#(x) = 1 - x^{\frac{1-\beta a}{2}} + o\big(x^{\frac{1-\beta a}{2}}\big). \tag{A.30}$$

It follows that

$$P_{k,\ell}(x) - P_\infty(x) = -\frac{x^{\frac{2-\beta a}{2}} + o\big(x^{\frac{2-\beta a}{2}}\big)}{\sqrt{x} + x^{1-\beta-\frac{1}{2}\beta a}(1+o(1))}. \tag{A.31}$$

There are again two cases to consider. If

$$\tfrac{1}{2} \leqslant 1 - \beta - \tfrac{1}{2}\beta a, \tag{A.32}$$

then

$$P_{k,\ell}(x) = P_\infty(x) - cx^{\frac{1}{2}(1-\beta a)} + o\big(x^{\frac{1}{2}(1-\beta a)}\big) \tag{A.33}$$

and $c = \frac{1}{2}$ if equality holds in (A.32) but $c = 1$ otherwise. If $\frac{1}{2} > 1 - \beta - \frac{1}{2}\beta a$, then

$$P_{k,\ell}(x) = P_\infty(x) - x^\beta + o(x^\beta), \tag{A.34}$$

i.e.,

$$P_{k,\ell}(x) = P_\infty(x) - cx^\delta + o(x^\delta), \tag{A.35}$$

where $\delta = \max\big\{\frac{1}{2}(1-\beta a), \beta\big\}$.

Appendix B

In this appendix, we show that for $L\sqrt{x}$ small

$$P_{*L}(x) = 1 - \frac{x^{\frac{1}{4}}}{\sqrt{L}} + O(\sqrt{x}), \tag{B.1}$$

and hence, for $L = [x^{-\beta}]$ with $\beta < \frac{1}{2}$, we obtain

$$P_{*L}(x) = 1 - x^{\frac{1}{4}+\frac{1}{2}\beta} + o\big(x^{\frac{1}{4}+\frac{1}{2}\beta}\big) \tag{B.2}$$

which is the result needed in sections 3 and 6.

Let R_L denote the generating function for first return to the root for walks that do not move beyond the $(L-1)$th vertex on the spine, i.e., they do not reach the vertex where the first tooth appears. Let Γ_L denote the two-point function defined by the sum over all walks from the root to L which do not return to the root and stop the first time they meet L, i.e., $\Gamma_L(x) = G^0_C(x; L)$ where $C = N_\infty$. Then by decomposing the walks that contribute to P_{*L} we obtain the equation

$$P_{*L} = R_L + \frac{\Gamma_L^2}{3 - P_\infty - P_{*L} - R_L}. \tag{B.3}$$

It is straightforward to solve this to obtain

$$P_{*L} = \frac{3 - P_\infty}{2} \pm \frac{1}{2}\sqrt{(3 - P_\infty)^2 - 4\big(\Gamma_L^2 + R_L(3 - P_\infty - R_L)\big)}. \tag{B.4}$$

Since $P_{*L}(x) \leqslant 1$ we must choose the $-$ sign in (B.4). The calculation of R_L is similar to that of P_ℓ in appendix A and gives

$$R_L(x) = (1 - x)\frac{(1+\sqrt{x})^{L-1} - (1-\sqrt{x})^{L-1}}{(1+\sqrt{x})^{L} - (1-\sqrt{x})^{L}} \tag{B.5}$$

so in particular

$$R_L(x) = \frac{L-1}{L} + O(xL). \tag{B.6}$$

Using an argument similar to the one leading to (24) we see that

$$\Gamma_L(x) = (1 - x)^{-\frac{1}{2}L+1} R_L(x) R_{L-1}(x) \cdots R_2(x) \tag{B.7}$$

for $L \geqslant 2$. If $L = 1$, then the product of the R factors in (B.7) is equal to 1 by definition. Rearranging we find

$$\Gamma_L(x) = (1 - x)^{L/2} \frac{2\sqrt{x}}{(1+\sqrt{x})^{L} - (1-\sqrt{x})^{L}} \tag{B.8}$$

so

$$\Gamma_L = \frac{1}{L} + O(Lx). \tag{B.9}$$

Noting that the expression under the square root in (B.4) can be written as

$$(3 - P_\infty - 2R_L + 2\Gamma_L)(3 - P_\infty - 2R_L - 2\Gamma_L), \tag{B.10}$$

we obtain (B.1) by inserting (11), (B.5) and (B.9) into (B.4).

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